Sparse Penalties in Dynamical System Estimation

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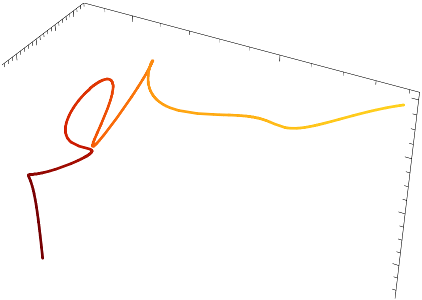
Dynamical Systems

Dynamical Systems



Systems Equations

- Dynamical system:

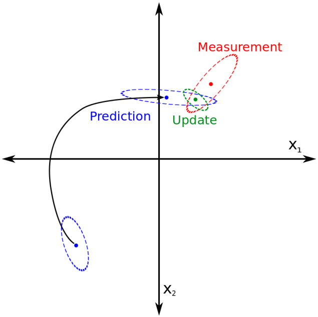
$$\boldsymbol{x}_n = f_n(\boldsymbol{x}_{n-1}) + \boldsymbol{\nu}_n$$

$$\boldsymbol{y}_n = \boldsymbol{G}_n \boldsymbol{x}_n + \boldsymbol{\epsilon}_n$$

- Estimate $\boldsymbol{x}_n$ at each iteration
- Like Kalman Filters, but with sparsity
- State, Innovations, or both can be sparse

Kalman Filter

- Linear and Gaussian assumptions $\rightarrow$ Kalman Filter is optimal
  - Estimates *best* $\boldsymbol{x}_n$ at time n
  - Uses only $\boldsymbol{y}_n$, $\hat{\boldsymbol{x}}_{n-1}$ and a parameter matrix (Cov[$\hat{\boldsymbol{x}}_{n-1}$])
  - Optimal AND Fast



Kalman as a Local Optimization

- Kalman really solves:

$$\{\hat{\boldsymbol{x}}_k\}_{k=0}^{n} = \arg\min_{\{\boldsymbol{x}_k\}} \left[\sum_{k=0}^{n} \|\boldsymbol{y}_k - \boldsymbol{G}_k \boldsymbol{x}_k\|_{\boldsymbol{Q}^{-1},2}^2 + \sum_{k=1}^{n} \|\boldsymbol{x}_k - \boldsymbol{F}_k \boldsymbol{x}_{k-1}\|_{\boldsymbol{R}^{-1},2}^2 \right]$$

- Where $\boldsymbol{Q} = \mathrm{Cov}[\epsilon]$ and $\boldsymbol{R} = \mathrm{Cov}[\boldsymbol{\nu}]$
- Calculates locally via

$$\hat{\boldsymbol{x}}_n = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y}_n - \boldsymbol{G}_n \boldsymbol{x}_n\|_{\boldsymbol{Q}^{-1},2}^2 + \|\boldsymbol{x} - \boldsymbol{F}_n \hat{\boldsymbol{x}}_{n-1}\|_{\boldsymbol{P}_{n|n-1}^{-1},2}^2 \right]$$

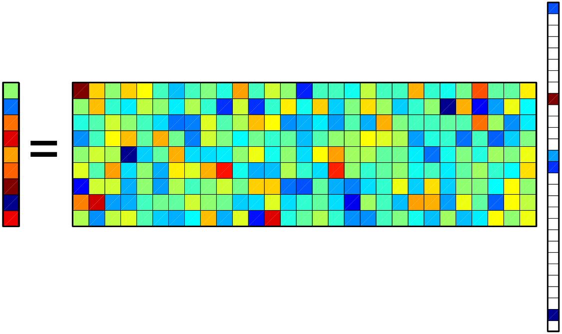
- Where $\boldsymbol{P}_{n|n-1} = \mathrm{Cov}[\boldsymbol{F}_n \hat{\boldsymbol{x}}_{n-1} + \boldsymbol{\nu}_n]$

Kalman Extensions

- No linearity/Gaussianity → Kalman optimality not guaranteed
- Extensions patch the process
  - Unscented/Extended Kalman → nonlinear processes
  - Robust Kalman → different noise statistics
- What about special structures?
- Can compressive sensing help with dynamic signal estimation?

Compressive Sensing and Sparse Coding

Compressive Sensing and Sparse Coding



Formalities

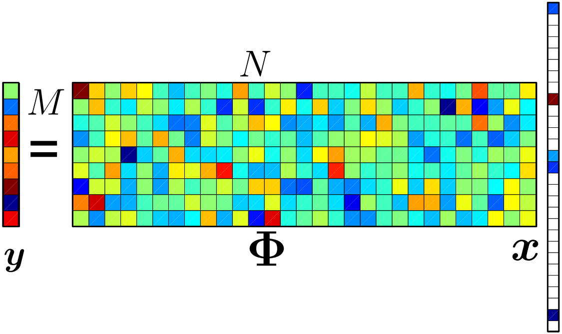
- Measurement: $\boldsymbol{y} \in \mathbb{R}^M$
- can write as

$$\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{\epsilon}$$

- $\boldsymbol{x} \in \mathbb{R}^N$ (typically $M < N$)
- Overcomplete so choice in $\boldsymbol{x}$: Choose sparse $\boldsymbol{x}$

Less Formal

- Can visualize as:



Finding the Coefficients

- Optimize ℓ_0 regularized least squares

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2 + \lambda \|\boldsymbol{x}\|_0 \right]$$

- Wonderful result: can optimize ℓ_1 regularized least squares instead

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2 + \lambda \|\boldsymbol{x}\|_1 \right]$$

- Now it's convex! (fast solvers)

Ties to Dynamic Signal Estimation

- Gives a framework where sparsity knowledge can be leveraged
- Just add ℓ_1
- Can this help sparse dynamics signals?

Previous Work

- Mostly based around separating support from values

  (Vaswani 2008, Ziniel et al. 2010)

  - ℓ_1 should get both simultaneously
- Some attempts to modify Kalman directly

  (Carmi et al. 2010)

  - Not the same problem: solution may look very different
- Smoothing algorithms

  (Angelosante et al. 2010)
- Homotopy updates for fast estimation

  (Asif et al. 2010)

  - Deals with one possibility of sparsity

Problem Statement

- Use compressive sensing ideas to reduce M for dynamical systems
  - *Where* can sparsity be leveraged
  - *How* can we leverage it (State? Innovations?)
- Want: a framework where ℓ_1 norms can represent sparse priors

Where is the Sparsity?

- 3 models:

$$\boldsymbol{x}_n = f_n(\boldsymbol{x}_{n-1}) + \boldsymbol{\nu}_n$$

- Model 1: Sparse states
- Model 2: Sparse innovations
- Model 3: Both sparse states and innovations
- No longer globally optimal

State Sparsity Term

- If the state is sparse: regularize with ℓ_1

$$\hat{\boldsymbol{x}}_n = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y}_n - \boldsymbol{G}_n \boldsymbol{x}_n\|_2^2 + \lambda_1 \|\boldsymbol{x}\|_1 + \lambda_2 \|\boldsymbol{x} - f_n(\hat{\boldsymbol{x}}_{n-1})\|_2^2 \right]$$

- Tracking a sparse number of objects

Innovations Sparsity Term

- If the Innovations is sparse: change second norm to ℓ_1

$$\hat{\boldsymbol{x}}_n = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y}_n - \boldsymbol{G}_n \boldsymbol{x}_n\|_2^2 + \lambda \|\boldsymbol{x} - f_n(\hat{\boldsymbol{x}}_{n-1})\|_1 \right]$$

- Tracking many objects with sparse appearances/disappearances

Sparsity Everywhere

- If both are sparse: regularize with two ℓ_1 norms

$$\hat{\boldsymbol{x}}_n = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y}_n - \boldsymbol{G}_n \boldsymbol{x}_n\|_2^2 + \lambda_1 \|\boldsymbol{x}\|_1 + \lambda_2 \|\boldsymbol{x} - f_n(\hat{\boldsymbol{x}}_{n-1})\|_1 \right]$$

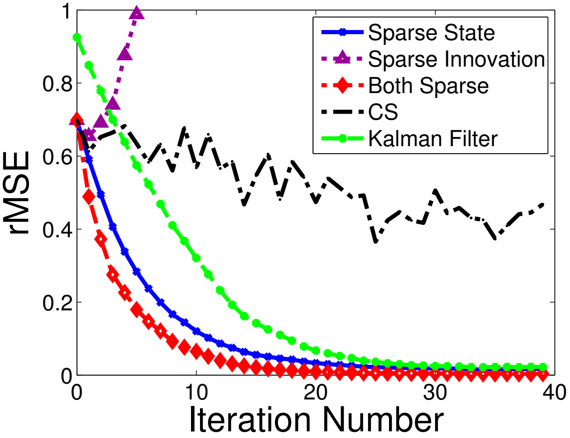
- Video/Dynamic MRI

Simulation Model

- Simulated data: $\boldsymbol{x}_n$ has length 500
- $\boldsymbol{F}_n$ is randomly selected permutation matrix with a random scaling (500x500 matrix)
- $\boldsymbol{G}_n$ is a random Gaussian matrix of size Mx500
- 3 Simulations:
  - Sparse states test: $\boldsymbol{x}_n$ is $K = 20$ sparse, $M = 30$ measurements per iteration
  - Sparse innovations test: $\boldsymbol{x}_n$ is dense, $\boldsymbol{\nu}_n$ is 30 sparse, $M = 200$ measurements per iteration
  - Both sparse states and innovations test: $\boldsymbol{x}_n$ is $K = 20$ sparse, sweep M and number of support changes P

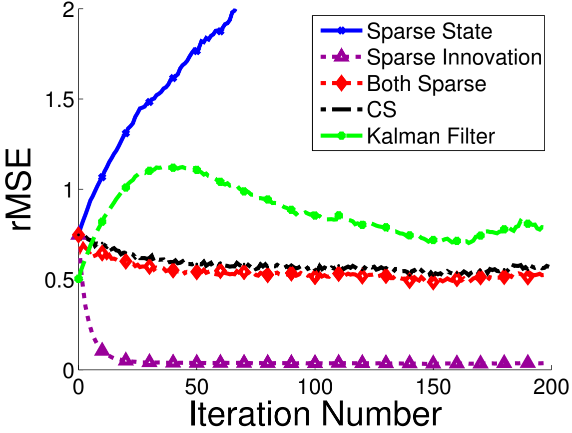
Sparsity in the State

- Sparse state/Gaussian innovations: short term gains



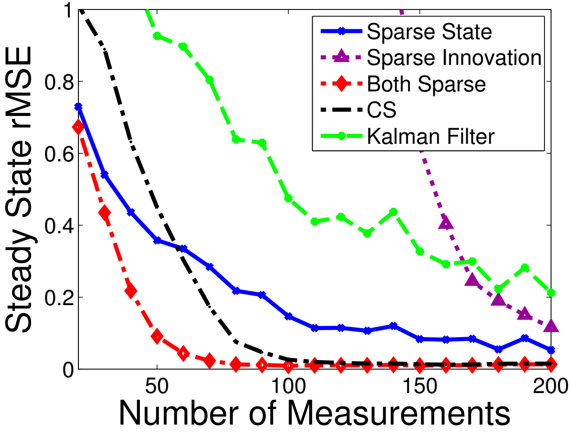
Sparsity in the Innovations

- Sparse Innovations: Tracks dense signals



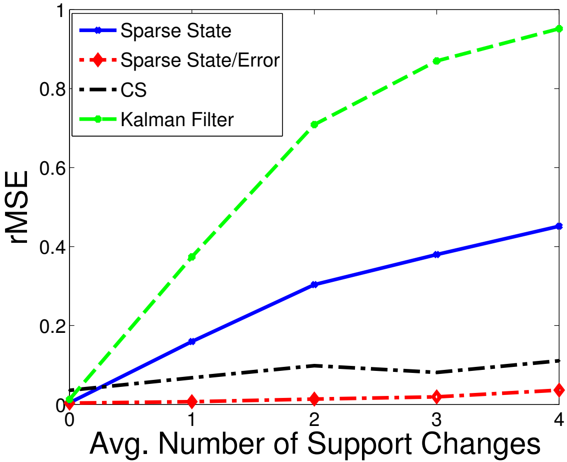
Sparsity in the Both

- Needs many less measurements/iteration



Robustness to Support Changes

- Robust to more changes in support



Summary

- Writing Dynamic state estimation as an optimization problem leads to a principled method of incorporating sparsity information
- Looked at 3 types of sparsity in dynamical system
- Determined a one-step update
- Simulated results show benefits in steady state rMSE, convergence time and rMSE per measurements

Future Work

- Show convergence
- Apply to real datasets (e.g. Dynamic MRI)
- Expand to propagate higher order statistics

Questions

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